Mathematical Economics Seminar I

Indivisible Goods Allocation[1]

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1 Common ownership economy

House allocation problem

- A *house allocation problem* is a triple $(A, H, \succ)$ where
  - $A = \{1, 2, \dots, n\}$ is a set of agents;
  - $H = \{h_1, h_2, \dots, h_n\}$ is a set of houses (or offices, parking spaces, dormitory rooms, school seats, course seat);
  - $\succ = (\succ_i)_{i \in A}$ is a preference profile, where $\succ_i$ is a strict preference of agent $i \in A$. Let $\succsim_i$ be the weak-preference relation associated with $\succ_i$.
- Real-life applications:
  - Organ allocation (deceased donor waiting list)
  - Dormitory room allocation at universities
  - Parking space or office allocation at workplaces.
- Given A and H, a problem is only denoted through the preference profile $\succ$.
- Solution of a house allocation problem is a matching:

$$\mu : A \to H$$

  μ is a one-to-one and onto function.

[1]This note is prepared for the class at University of Seoul. This note may have some typos and errors. If you find any error or typo, please let me know via e-mail (chomhmh@uos.ac.kr).

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Given a matching μ, house $\mu_i \equiv \mu(i)$ is the assigned house of agent i a under matching μ.

Given (A, H), let $\mathcal{M}(A, H)$ be the set of matchings.

Sometimes, we write $\mathcal{M}$ for $\mathcal{M}(A, H)$, if there is no confusion.

- A matching μ is *Pareto efficient*, if there is no other matching ν such that $\nu_i \succsim_i \mu_i$ for all $i \in A$ and $\nu_i \succ_i \mu_i$ for some $i \in A$.

- A (*deterministic*) *mechanism* ϕ is a procedure that assigns a matching for each house allocation problem.

  A mechanism ϕ can be represented as a function, which maps each housing market $(A, H, \succ)$ to a matching $\phi(A, H, \succ) \in \mathcal{M}(A, H)$.

  With a slight abuse of notation, given A and H, ϕ can be represented a function that maps each preference profile $\succ$ to a matching $\phi(\succ) \in \mathcal{M}$.

- For notational convenience, for a mechanism ϕ, $\phi_i(\succ)$ denotes agent i's allocation of house under the matching $\phi(\succ)$. That is, $\phi_i(\succ) \equiv \phi(\succ)(i)$.

- A mechanism ϕ is *strategyproof* (or, (*dominant strategy*) *incentive compatible*), if for any profile $\succ$, there is no agent $i \in A$ and no preference relation $\succ'_i$ such that

$$\phi_i(\succ'_i, \succ_{-i}) \succ_i \phi_i(\succ_i, \succ_{-i}).$$

- Given $\succ_i$ and $h \in H$, $\succ'_i$ is a *push-up of* $\succ_i$ *for* h, if for all $h' \in H$, $h' \succ'_i h$ implies $h' \succ_i h$.

- Consider a strategyproof mechanism ϕ. If $\succ'_i$ is a push-up of $\succ_i$ for $\phi_i(\succ)$, then $\phi_i(\succ) = \phi_i(\succ'_i, \succ_{-i})$.

- A mechanism is *Pareto efficient*, if it always assigns a Pareto efficient matching for each preference profile reported.

Serial dictatorship mechanism

- A priority ordering is a function $f : \{1, 2, \ldots, n\} \to A$ that is one-to-one and onto.

  $f(k)$ is the agent with the kth highest priority under f.

Let $\mathcal{F}$ be the set of orderings.

- A serial dictatorship is defined through a priority ordering of agents.
- Let π^f be the serial dictatorship induced by priority ordering f.
- The matching $\pi^f(\succ)$ is found as follows iteratively: The first agent $f(1)$ gets her first choice, the second agent $f(2)$ gets her first choice excluding the house assigned to the first agent,..., kth agent $f(k)$ gets her first choice excluding the houses assigned to the houses all agents before $f(k)$.

Theorem 1. *A serial dictatorship π^f is strategyproof.*

Proof. Let f be a priority ordering and π^f be the induced serial dictatorship. We prove the theorem by iteration. The first agent $f(1)$ cannot do better than reporting any other preferences, since she already receives her first choice house under her reported preferences, ..., $f(k)$ cannot do better than reporting her true preferences, since the houses distributed until $f(k)$ is independent of $f(k)$'s preferences and $f(k)$ receives her first choice among the remaining houses given her reported preferences. $\square$

Theorem 2. *A serial dictatorship π^f is Pareto efficient.*

Proof. Suppose that $\mu = \pi^f(\succ)$ is the outcome of a serial dictatorship π^f for preference profile $\succ$. We prove the theorem by contradiction. Suppose that the serial dictatorship is not Pareto efficient. There exists a matching ν that Pareto-dominates μ under reported preferences $\succ$. In particular, there exists some agent $i \in A$ such that $\nu_i \succ_i \mu_i$, in particular, let i be the highest priority agent in f with this property. Let $f(k) \equiv i$. Since for any other agent $j \in A \backslash \{i\}$, $\nu_j \succsim_j \mu_j$, for any agent $f(l)$ with $l < k$ (that is $f(l)$ has higher priority than $f(k) \equiv i$), we have $\nu(f(l)) = \mu(f(l))$. Therefore, in serial dictatorship π^f, when it is $f(k)$'s turn to choose, houses ν_i and μ_i are still available. However, she chooses $\mu_i = \pi_i^f(\succ)$ in the serial dictatorship contradicting $\nu_i \succ_i \mu_i$. So there is no matching that can Pareto-dominate μ. Hence, $\mu = \pi^f(\succ)$ is Pareto efficient. $\square$

- A relabeling r is a permutation of houses.

  For any house $h \in H$, h is called $r(h)$ under relabeling r.
- Let $\succ^r$ be the preference profile under which each house h is replaced with its new name $r(h)$.

- A mechanism ϕ is *neutral*, if when we relabel the houses (rename the houses), the mechanism assigns each agent the relabeled version of the old house that she was assigned.

  Formally, a mechanism ϕ is *neutral*, if for any relabeling r and any preference profile $\succ$, we have
  $$\phi_i(\succ^r) = r(\phi_i(\succ))$$
  for any agent $i \in A$.

- It is straightforward to see that a serial dictatorship π^f is neutral.

- A mechanism is *non-bossy*, if the outcome of the mechanism remains unchanged when an agent's assigned house remains the same even though she manipulates her preferences.

  Formally, a mechanism ϕ is *non-bossy*, if for any $i \in A$, for any $\succ_i, \succ'_i$ and $\succ_{-i}$,
  $$\phi_i(\succ'_i, \succ_{-i}) = \phi_i(\succ_i, \succ_{-i}) \implies \phi(\succ'_i, \succ_{-i}) = \phi(\succ_i, \succ_{-i})$$

Theorem 3. *A serial dictatorship π^f is non-bossy.*

Proof. Let f be a priority ordering and $\succ$ be a preference profile. We prove the theorem by iteration.

Consider $f(1)$. Suppose that she changes her preferences to $\succ'_{f(1)}$ and still obtains house $\pi^f_{f(1)}(\succ_{f(1)}, \succ_{-f(1)})$. Since the remaining houses and the preferences are the same under both $(\succ_{f(1)}, \succ_{-f(1)})$ and $(\succ'_{f(1)}, \succ_{-f(1)})$, the outcome does not change.

Now consider $f(k)$. First of all, notice that she cannot change the distributed houses to the agents ordered before her by changing her preferences. Second, using the same idea, if she obtains the house
$$\pi^f_{f(k)}(\succ_{f(k)}, \succ_{-f(k)})$$
by changing her preferences to $\succ'_{f(k)}$, we have
$$\begin{array}{rl} & H \setminus \cup_{i=1}^k \{\pi^f_{f(k)}(\succ'_{f(i)}, \succ_{-f(i)})\} \\ = & H \setminus \cup_{i=1}^k \{\pi^f_{f(k)}(\succ_{f(i)}, \succ_{-f(i)})\} \end{array}.$$

Therefore, since the other agents have the same preferences, the agents who are ordered after $f(k)$ also get the same houses. Hence, the outcome of the mechanism does not change. $\square$

- A mechanism ϕ is *group-strategyproof* (or, (*strongly*) *group incentive compatible*), if there is no set $B \subseteq A$ and preferences $\succ'_B, \succ_B$ and $\succ_{-B}$ such that for every agent $i \in B$,

$$\phi_i(\succ'_B, \succ_{-B}) \succsim_i \phi_i(\succ_B, \succ_{-B})$$

and for some agent $i \in B$,

$$\phi_i(\succ'_B, \succ_{-B}) \succ_i \phi_i(\succ_B, \succ_{-B}).$$

Theorem 4 (Pápai (2000))**.** *A mechanism ϕ is group-strategyproof if and only if it is non-bossy and strategyproof.*

Proof. ($\Rightarrow$) It is obvious that group-strategyproofness implies strategyproofness. To see that group-strategyproofness implies non-bossiness, suppose that ϕ is not non-bossy. Then, for some $i, i' \in A$, and for some $\succ'_i$ and $\succ_i$,

$$\phi_i(\succ'_i, \succ_{-i}) = \phi_i(\succ_i, \succ_{-i})$$

but

$$\phi_{i'}(\succ'_i, \succ_{-i}) \neq \phi_{i'}(\succ_i, \succ_{-i}).$$

This means that $\phi_{i'}(\succ'_i, \succ_{-i}) \succ_{i'} \phi_{i'}(\succ_i, \succ_{-i})$ or $\phi_{i'}(\succ_i, \succ_{-i}) \succ_{i'} \phi_{i'}(\succ'_i, \succ_{-i})$ holds. If $\phi_{i'}(\succ'_i, \succ_{-i}) \succ_{i'} \phi_{i'}(\succ_i, \succ_{-i})$, $B = \{i, i'\}$ with $(\succ_i, \succ_{-i})$ has an incentive to misrepresent their preferences as $(\succ'_i, \succ_{-i})$. If $\phi_{i'}(\succ_i, \succ_{-i}) \succ_{a'} \phi_{i'}(\succ'_a, \succ_{-i})$, $B = \{i, i'\}$ with $(\succ'_i, \succ_{-i})$ has an incentive to misrepresent their preferences as $(\succ_i, \succ_{-i})$.

($\Leftarrow$) Suppose that ϕ is not group-strategyproof. Then, there exists $B \subset A$ such that, for all $i \in B$,

$$\phi_i(\succ'_B, \succ_{-B}) \succsim_i \phi_i(\succ)$$

for some $\succ = (\succ_B, \succ_{-B})$ and $\succ'_B$. Without loss of generality, let $B = \{1, \ldots, m\}$. For each $i \in B$, let $\succ^o_i$ be agent i's preference that preserves the ordering of $\succ_i$ except that $\phi_i(\succ'_B, \succ_{-B})$ is on the top in her list. By strategyproofness,

$$\phi_1(\succ^o_1, \succ_{-1}) = \phi_1(\succ)$$

should hold. Then, by non-bossiness,

$$\phi(\succ_1^o, \succ_{-1}) = \phi(\succ).$$

Repeating the same argument for agents $2, \ldots, m$, we have $\phi(\succ_B^o, \succ_{-B}) = \phi(\succ)$. Now, note that for all $i \in B$, $\succ_i^o$ is a push-up of $\succ_i'$, for $\phi_i(\succ_B', \succ_{-B})$, so that $\phi(\succ_B^o, \succ_{-B}) = \phi(\succ_B', \succ_{-B})$, by strategyproofness and non-bossiness. Thus, $\phi(\succ_B', \succ_{-B}) = \phi(\succ)$, which implies that ϕ is group-strategyproof. $\square$

Theorem 5 (Svensson (1999))**.** *A mechanism is strategyproof, non-bossy and neutral if and only if it is a serial dictatorship.*

Proof. ($\Leftarrow$) This is proved in the above theorems.

($\Rightarrow$) Let ϕ be an strategyproof, non-bossy and neutral mechanism. We need to show that there exists a priority ordering f such that $\pi^f(\succ) = \phi(\succ)$ for any preference profile $\succ$. We construct f iteratively as follows: Let $\succ$ be a preference profile such that for all $i \in A$, $h_1 \succ_i h_2 \succ_i \ldots \succ_i h_n$. For each i, Let $f(i)$ be agent who receives h_i under $\phi(\succ)$. That is, $f(1)$ is the agent who receives h_1, $f(2)$ is the agent who receives h_2, and so on. Clearly, $\pi^f(\succ) = \phi(\succ)$.

Without loss of generality, let $1 = f(1), 2 = f(2), \ldots, n = f(n)$.

Let $\succ'$ be any preference profile. Let us relabel houses so that agent 1's first choice under $\succ'$ is now called h_1, agent 2's first choice in $H \backslash \{h_1\}$ under $\succ'$ is now called h_2, ..., agent k's first choice in $H \backslash \{h_1, \ldots, h_{k-1}\}$ under $\succ'$ is now called h_k....... Let r be this relabeling. Note that

$$\pi_k^f(\succ'^r) = h_k = \pi_k^f(\succ) \tag{1.1}$$

for any $k \in \{1, 2, \ldots, n\}$.

We will prove that $\phi_k(\succ'^r) = h_k$ by induction. Consider agent 1. By strategyproofness of ϕ, we have

$$\phi_1(\succ_1'^r, \succ_{-1}) \succsim_1'^r \phi_1(\succ) = h_1$$

implying that $\phi_1(\succ_1'^r, \succ_{-1}) = h_1 = \phi_1(\succ)$. By non-bossiness of ϕ, we have

$$\phi(\succ_1'^r, \succ_{-1}) = \phi(\succ).$$

Let $k > 1$. Assume that for any $i < k$, we have $\phi(\succ_{\{1,\ldots,i\}}'^r, \succ_{-\{1,\ldots,i\}}) = \phi(\succ)$. Consider agent k.

By strategyproofness of ϕ, we should have

$$\phi_k(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}) \succsim'^r_k \phi_k(\succ'^r_{\{1,\ldots,k-1\}}, \succ_{-\{1,\ldots,k-1\}}) = \phi_k(\succ) = h_k.$$

By construction of h_k, it holds that $\phi_k(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}) \in \{h_1, \ldots, h_k\}$. In addition, by strategyproofness of ϕ, we should have

$$h_k = \phi_k(\succ) = \phi_k[\succ'^r_{\{1,\ldots,k-1\}}, \succ_{-\{1,\ldots,k-1\}}] \succsim_k \phi_k(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}),$$

which implies $\phi_k(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}) \in \{h_k, \ldots, h_n\}$. Thus, we obtain that

$$\phi_k(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}) = h_k = \phi_k(\succ'^r_{\{1,\ldots,k-1\}}, \succ_{-\{1,\ldots,k-1\}}).$$

By non-bossiness of ϕ,

$$\phi(\succ'^r_{\{1,\ldots,k\}}, \succ_{-\{1,\ldots,k\}}) = \phi(\succ'^r_{\{1,\ldots,k-1\}}, \succ_{-\{1,\ldots,k-1\}}).$$

Now, we have

$$\phi(\succ'^r) = \phi(\succ) = \pi^f(\succ) \tag{1.2}$$

We have

$$\begin{array}{rcll} \phi(\succ') & = & r^{-1}(\phi(\succ'^r)) & \text{by neutrality of } \phi \\ & = & r^{-1}(\pi^f(\succ)) & \text{by (1.2)} \\ & = & r^{-1}(\pi^f(\succ'^r)) & \text{by (1.1)} \\ & = & \pi^f(\succ') & \text{by neutrality of } \pi^f \end{array}$$

□

Corollary 1. *Non-bossiness, strategyproofness and neutrality imply Pareto-efficiency.*

- A mechanism that is non-bossy, strategyproof, but not neutral:

  Let A and H be fixed. Let f be a priority ordering in which agent 1 is in the first place, agent 2 is in the second place, and g be a priority ordering in which agent 2 is in the first place and agent 1 is in the second place. Let the rest of the orderings f and g be the same. Mechanism $\tilde{\phi}$ is defined as follows: When h_1 is a 1's first choice $\tilde{\phi}$ chooses the matching selected by serial dictatorship π^f, and otherwise $\tilde{\phi}$ chooses the matching selected by serial dictatorship π^g .

- $\tilde{\phi}$ *is strategyproof:*

  Only agent 1 can change the order, so for all other agents the mechanism is a serial dictatorship. Hence, it is a dominant strategy to tell true preferences for all agents $i \in A \backslash \{1\}$. What about agent 1? She can only change the order by putting h_1 in the first place or not. Two cases:

  - If h_1 is her first choice, if she tells the truth, she will pick first and she will get h_1, so in this case she has no incentive to lie.
  - If h_1 is not her first choice, if lies and lists h_1 first, then she will get h_1, since she will pick first. When she tells the truth, in this case h_1 is her second choice or less. In this case, she is guaranteed to get at least her second choice, since she picks second. Therefore, the house she gets in this case is weakly better than h_1, concluding the proof.

- $\tilde{\phi}$ *is non-bossy:*

  Take $i \in A \backslash \{1\}$. Let $\succ_i$, $\succ_i'$, and $\succ_{-i}$ be such that $\tilde{\phi}_i(\succ_i', \succ_{-i}) = \tilde{\phi}_i(\succ)$. If agent i did not change the order by submitting $\succ_i'$ instead of $\succ_i$, clearly $\tilde{\phi}(\succ_i', \succ_{-i}) = \tilde{\phi}(\succ)$ by non-bossiness of serial dictatorship. Suppose $i = 1$ and she changes the order by submitting $\succ_i'$ instead of $\succ_i$. In both cases, she gets h_1. Therefore, agent 2 gets the same house under both matchings $\tilde{\phi}(\succ_i', \succ_{-i})$ and $\tilde{\phi}(\succ)$ implying that the other agents receive the same houses (since their rankings are the same under f and g). So, $\tilde{\phi}(\succ_i', \succ_{-i}) = \tilde{\phi}(\succ)$ concluding the proof.

- $\tilde{\phi}$ *is not neutral:*

  Consider a relabeling in which h_1 is relabeled as h_2 and h_2 is relabeled as h_1. And let agent 1 and agent 2 like h_1 as their first choice and like h_2 as their second choice under $\succ$. We have $\tilde{\phi}_1(\succ) = h_1$ but

  $$\tilde{\phi}_1(\succ^r) = h_1 = r(h_2) \neq r(\tilde{\phi}_1(\succ)) = h_2.$$

- **Homework**: Find a mechanism that is bossy, strategyproof and neutral.

- **Homework**: Find a mechanism that is non-bossy, non-strategyproof and neutral mechanisms.

- Theorem 4 means that, if a matching is obtained from a serial dictatorship, then it is Pareto efficient. One may be interested in the opposite direction.

Theorem 6 (Abdulkadiroğlu and Sönmez (1998))**.** *For every Pareto efficient matching in a house allocation problem, there is a serial dictatorship that achieves it.*

Proof. Let μ be a Pareto efficient matching. Given a matching, construct a binary relation on the agents as follows. For i and j, $j \succ_P i$ if and only if $\mu(j) \succ_i \mu(i)$.

Suppose that the binary relation $\succ_P$ has a cycle. Then, the agents involved in this cycle can be better off by exchanging their houses in matching μ. This contradicts that the μ is Pareto efficient. Thus, the binary relation $\succ_P$ does not have a cycle.

Then, we can easily construct a binary relation $\succ_R$ that is complete and transitive and agrees with $\succ_P$. This binary relation $\succ_R$ determines the priority ordering under which the serial dictatorship yields the matching μ. □

- At the first glance, the serial dictatorship seems undesirable since it discriminates between the agents.

  However, this undesirability can be easily handled by randomly choosing an ordering for the agents.

- A *random serial dictatorship* is a mechanism that randomly chooses an ordering with uniform distribution and then applies the induced simple serial dictatorship.

- A random serial dictatorship generates a *lottery mechanism* in the sense that the mechanism generates a lottery over the matchings.

- Suppose mechanisms are defined under variable populations and A be the global set of potential agents. A mechanism is *consistent*, if upon removal of agents and their assignments from a problem, its outcome for the remaining agents stays unchanged when the mechanism is re-executed in the remaining problem.

Theorem 7 (Ergin (2000))**.** *A mechanism is consistent, Pareto efficient and neutral if and only if it is a serial dictatorship.*

Corollary 2. *Consistency, Pareto efficiency and neutrality imply group-strategyproofness.*

2 Private ownership economy

Housing market

- A *housing market* is a list $(A, H, \succ)$.
  - $A = \{1, 2, \ldots, n\}$is a set of agents;
  - $H = \{h_1, h_2, \ldots, h_n\}$ is a set of houses with each h_i being owned by agent i;
  - $\succ = (\succ_i)_{i \in A}$ is a preference profile, where $\succ_i$ is a strict preference of agent i.

    Let $\succsim_i$ be the weak-preference relation associated with $\succ_i$.
- A housing market is a *non-transferable utility* (NTU) cooperative game.
- An outcome of a housing market is a *matching* μ, which is a one-to-one and onto function:

$$\mu : A \to H.$$

  $\mu_i \equiv \mu(i)$ has the same meaning as in the house allocation problems.

  Given (A, H), let $\mathcal{M}(A, H)$ be the set of matchings.
- A matching μ *is (*weakly*) blocked by a coalition* $B \subset A$, if there is another matching ν such that
  - for any $i \in B$, $\nu(i) = h_l$ for some $l \in B$, and
  - for any $i \in B$, $\nu(i) \succsim_i \mu(i)$ and for some $i \in B$, $\nu(i) \succ_i \mu(i)$.
- A matching μ is *individually rational*, if it is not blocked by $B = \{i\}$ for any $i \in A$.

  Any individually rational matching μ satisfies that for each $i \in A$, $\mu(i) \succsim_i h_i$.
- A matching μ is *Pareto efficient*, if it is not blocked by $B = A$.
- A (*strong*) *core* is the set of matchings μ that are not blocked by any coalition $B \subseteq A$.

  A *core matching* is a matching in the core.
- Any core matching is individually rational and Pareto efficient.

- A *mechanism* ϕ is also defined as the same way in house allocation problems. That is, a mechanism is a function that maps each housing market $(A, H, \succ)$ to a matching $\phi(A, H, \succ) \in \mathcal{M}(A, H)$.

  Again, given that A and H are fixed, ϕ can be represented a function that maps each preference profile $\succ$ to a matching $\phi(\succ) \in \mathcal{M}$.

  For natational convenience, for a mechanism ϕ, $\phi_i(\succ)$ denotes agent i's allocation of house under the matching $\phi(\succ)$. That is, $\phi_i(\succ) \equiv \phi(\succ)(i)$.

- For mechanisms, *Pareto efficiency*, *strategyproofness*, and *group-strategyproofness* have the same definitions as in house allocation problems.

- A mechanism ϕ is *Pareto efficient*, if it always assigns a Pareto efficient matching for each preference profile.

- A mechanism ϕ is *strategyproof*, if for any profile $\succ$, there is no agent $i \in A$ and no preference relation $\succ_i'$ such that

$$\phi_i(\succ_i', \succ_{-i}) \succ_i \phi_i(\succ_i, \succ_{-i}).$$

- A mechanism ϕ is *group-strategyproof*, if there is no set $B \subseteq A$ and preferences $\succ_B', \succ_B$ and $\succ_{-B}$ such that for every agent $i \in B$,

$$\phi_i(\succ_B', \succ_{-B}) \succsim_i \phi_i(\succ_B, \succ_{-B})$$

  and for some agent $i \in B$,

$$\phi_i(\succ_B', \succ_{-B}) \succ_i \phi_i(\succ_B, \succ_{-B}).$$

Gales top-trading cycles (TTC) algorithm

- Consider an iterative algorithm over a directed graph with houses and agents as the nodes, which constructs a matching as follows in several steps.

- Step 1: Let each agent point to the owner of her most preferred house. Then, there is necessarily a cycle and no two cycles intersect (since preferences are strict). Remove all cycles from the problem by assigning each agent the house whose owner she is pointing to.

$\vdots$

- <u>Step k</u>: Let each remaining agent point to the owner of her most preferred house among the remaining houses. Again, there is necessarily a cycle and no two cycles intersect. Remove all cycles from the problem by assigning each agent the house whose owner she is pointing to.

$\vdots$

- Continuing this procedure until all agents are assigned a house and removed from the problem.

Example 1 (Gale's Top Trading Cycles Algorithm)**.** Let $A = \{1, 2, \ldots, 16\}$ with each i owns a house h_i. Let the preference profile $\succ$ be given as

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h_{15}	h_3	h_1	h_2	h_9	h_6	h_6	h_6	h_{11}	h_7	h_2	h_4	h_6	h_8	h_1	h_5
$\vdots$	h_4	h_3	$\vdots$	$\vdots$	$\vdots$	h_7	h_{12}	$\vdots$	h_3	h_4	h_{14}	h_{13}	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$				$\vdots$	$\vdots$		h_{12}	h_{16}	$\vdots$	$\vdots$			
									h_{10}						
									$\vdots$						

Step 1: $1 - h_{15}$; $15 - h_1$; $6 - h_6$
Step 2: $3 - h_3$; $13 - h_{13}$; $7 - h_7$
Step 3: $2 - h_4$; $4 - h_2$
Step 4: $16 - h_5$; $5 - h_9$; $9 - h_{11}$; $11 - h_{16}$; $12 - h_{14}$; $14 - h_8$; $8 - h_{12}$
Step 5: $10 - h_{10}$

Outcome:

$$\psi(\succ) = \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ h_{15} & h_4 & h_3 & h_2 & h_9 & h_6 & h_7 & h_{12} & h_{11} & h_{10} & h_{16} & h_{14} & h_{13} & h_8 & h_1 & h_5 \end{array}$$

$\diamondsuit$

- It turns out that the core has a unique matching $\bar{\mu}$ and Gale's TTC algorithm results in this matching $\bar{\mu}$.

The matching in the core is called a *core matching*

The mechanism that coincides with Gale's TTC algorithm is called a *core mechanism.*

In this section, we denote the core mechanism by $\bar{\phi}$.

Theorem 8 (Shapley and Scarf (1974), Roth and Postlewaite (1977))**.** *In every housing market, the core is nonempty and it is a singleton.*

Proof. Let μ be the matching obtained as the result of Gale's TTC algorithm.

We first show that μ is in the core. Suppose that μ is blocked by a coalition B via some matching ν. Let $C_1, \ldots, C_K$ be the sets of agents in the cycles (in the order they are removed) in Gale's TTC algorithm. That is, C_k is the set of agents who are removed in step k in Gale's TTC algorithm. Note that no agent in C_1 can be made better off over μ. If $C_1 \cap B \neq \varnothing$, then $C_1 \subset B$. Since each agent in C_1 should get their first choice under μ, they would get each other's endowment under ν and would be indifferent with respect to μ. Then, $B \backslash C_1$ blocks μ via ν as well. Consider $B \backslash C_1$ and ν. No agent in $C_2 \cap (B \backslash C_1)$ can be made better off. If $C_2 \cap (B \backslash C_1) \neq \varnothing$, then $C_2 \subset B$. Then, $B \backslash (C_1 \cup C_2)$ blocks μ via ν. Iteratively, we can continue and finally reach a contradiction that $\varnothing$ blocks μ via ν.

Next, we want to show that there is no other matching in the core. Consider a matching $\nu \neq \mu$. Let agent j be the first agent who satisfies $\nu(j) \neq \mu(j)$ during Gale's TTC algorithm (according to the order of the cycles $C_1, \ldots, C_K$, and if there are multiple agents in a cycle like j, choose one of them arbitrarily). Let j be in cycle C_k. For every agent i assigned before the cycle C_k, $\nu(i) = \mu(i)$ holds. Given this, for every agent $i \in C_k$, $\mu(i) \succsim_i \nu(i)$ for all $i \in C_k$. Then, we have $\mu(j) \succ_j \nu(j)$ by strictness of preferences. In addition, for each agent $i \in C_k$, $\mu(i) = h_m$ for some $m \in C_k$ by construction of μ and C_k. Hence, C_k blocks ν via μ. This completes the proof. $\square$

Theorem 9 (Roth (1982))**.** *The core mechanism $\bar{\phi}$ is strategyproof.*

Proof. Let $\bar{\phi}$ be the core mechanism. Let $\succ$ be a preference profile. Let $C_1, \ldots, C_k, \ldots C_K$ be the sets of agents removed in each step k in Gale's TTC mechanism in the construction of $\mu = \bar{\phi}(\succ)$. We prove the theorem by iteration on cycles.

Each agent in C_1 receives her first choice under μ with respect to $\succ$. So, none of these agents will benefit by reporting a different preference relation. Moreover, observe that C_1 will form as it is in Step 1, regardless of any agent in $A \backslash C_1$ submits a different preference relation.

$\vdots$

Each agent in C_k receives her first choice under μ in $H\setminus \cup_{i=1}^{k-1} \{\mu(C_i)\}$ with respect to $\succ$. Since the previous cycles are unaffected by them reporting different preferences, if one agent changes her preferences still the same houses will be assigned to the agent in $\cup_{i=1}^{k-1} C_i$. Therefore, this agent will not benefit by misreporting her preferences under Gale's TTC algorithm. This completes the proof. $\square$

Theorem 10 (Ma (1994))**.** *A mechanism is strategyproof, Pareto efficient, and individually rational if and only if it is the core mechanism $\bar{\phi}$.*

Proof. ($\Leftarrow$) This part is already proven.

($\Rightarrow$) Let ϕ be a strategyproof, Pareto efficient, and individually rational mechanism. Let $\succ$ be an arbitrary preference profile. We will first create a preference profile $\succ'$ from $\succ$. Then, by showing $\phi(\succ') = \bar{\phi}(\succ')$ and $\phi(\succ) = \phi(\succ')$, we will completes the proof.

Let $\bar{\mu} = \bar{\phi}(\succ)$. Since $\bar{\mu}$ is individually rational, $\bar{\mu}(l) \succsim_l h_l$ for each agent $l \in A$. For each l, define $\succ'_l$ as follows:

- If $\bar{\mu}(l) \neq h_l$, then squeeze h_l just below $\bar{\mu}(l)$ (i.e., $\bar{\mu}(l)$ is the last acceptable house) and do not change the relative ordering of other houses. We have $h \succ'_l h'$ if and only if $h \succ_l h'$ for any $h, h' \in H\setminus\{h_l\}$ and $h_l \succ'_l h$ if and only if $\bar{\mu}(l) \succ'_l h$ for any $h \in H\setminus\{h_l\}$.

- If $\bar{\mu}(l) = h_l$, then $\succ'_l = \succ_l$.

Observe that this operation does not change the core. That is, $\bar{\phi}(\succ') = \bar{\phi}(\succ) = \bar{\mu}$.

<u>Claim 1</u>: $\phi(\succ') = \bar{\phi}(\succ')$.

Let $\mu' = \phi(\succ')$. Let $C_1, C_2, ..., C_k$ be the sets of agents removed in the cycles during Gale's TTC algorithm for $\succ$. We prove Claim 1 by iteration.

Suppose that for some agent $l \in C_1$, $\mu'(l) \neq \bar{\mu}(l)$. By individual rationality of ϕ, we have $\mu'(l) = h_l$. But by individual rationality of ϕ, we have $\mu'(i) = h_i$ for each $i \in C_1$. Now instead of all these agents getting their endowments (which are their second choices under $\succ'$), if they got their first choices, and rest of the matching μ' did not change, this new matching Pareto-dominates μ' under $\succ'$. This contradicts that ϕ is PE under $\succ'$. This shows that for all $i \in C_1$, $\mu'(i) = \bar{\mu}(i)$ holds.

Suppose that for some agent $l \in C_2$ we have $\mu'(l) \neq \bar{\mu}(l)$. Since all the better houses than $\bar{\mu}(l)$ are distributed to agents in C_l under μ'. By a similar argument then by IR of ϕ, we have $\mu'(i) = h_i$ for every $i \in C_2$. Instead of all these agents getting their endowments (which are ranked lower than their assignments under $\bar{\mu}$ under $\succ'$), we can assign them their assignments under $\bar{\mu}$ keeping the rest of the matching μ' intact, this new matching Pareto-dominates μ', contradicting ϕ is PE under $\succ'$.

We showed that for any agent $i \in C_2$, $\mu'(i) = \bar{\mu}(i)$ holds. Continuing this procedure proves Claim 1.

Note that Claim 1 also shows that $\phi(\succ')$ is the unique Pareto-efficient and individually rational matching under $\succ'$.

Claim 2: $\phi(\succ) = \phi(\succ')$.

We prove Claim 2 by induction. Let $B = \{i\} \subseteq A$. By SP of ϕ,

$$\begin{aligned} \phi_i(\succ_i, \succ'_{-i}) &\succsim_i \phi_i(\succ') \\ \phi_i(\succ') &\succsim'_i \phi_i(\succ_i, \succ'_{-i}) \end{aligned}$$

Since ϕ satisfies IR, the above relation implies that

$$\phi_i(\succ_i, \succ'_{-i}) = \phi_i(\succ')$$

Therefore, while problems $(\succ_i, \succ'_{-i})$ and $\succ'$ differ in preferences of agent i, her assignment under ϕ does not differ in these two problems. Hence matching $\phi(\succ_i, \succ'_{-i})$ not only has to be Pareto efficient and individually rational under $(\succ_i, \succ'_{-i})$ but also under $\succ'$ and therefore by the fact that $\phi(\succ') = \bar{\phi}(\succ')$ and $\bar{\phi}(\succ')$ is the unique Pareto efficient and individually rational matching under $\succ'$, we have

$$\phi(\succ_i, \succ'_{-i}) = \phi(\succ')$$

Let $k > 1$. In the inductive step, assume that for any $B \subseteq A$ with $|B| \leq k$,

$$\phi(\succ_B, \succ'_{-B}) = \phi(\succ'). \tag{2.1}$$

Fix $B \subseteq A$ such that $|B| = k + 1$. Fix $i \in B$. By strategyproofness of ϕ, we have

$$\begin{aligned} \phi_i(\succ_B, \succ'_{-B}) &\succsim_i \phi_i(\succ_{B\setminus\{i\}}, \succ'_{-(B\setminus\{i\})}) \\ \phi_i(\succ_{B\setminus\{i\}}, \succ'_{-(B\setminus\{i\})}) &\succsim'_i \phi_i(\succ_B, \succ'_{-B}) \end{aligned}$$

The above relation and the construction of $\succsim_i'$ imply that

$$\phi_i(\succ_B, \succ'_{-B}) \;=\; \phi_i(\succ_{B\backslash\{i\}}, \succ'_{-(B\backslash\{i\})}) \;=\; \phi_i(\succ'), \tag{2.2}$$

where the second equality follows from the inductive assumption (2.1) (since $|B\backslash\{i\}| = k$). Since the choice of $i \in B$ is arbitrary, (2.2) holds for all $i \in B$. Therefore, while problems $(\succ_B, \succ'_{-B})$ and $\succ'$ differ in preferences of agents in B, there assignments under ϕ do not differ in these two problems. Hence, matching $\phi(\succ_B, \succ'_{-B})$ not only has to be Pareto efficient and individually rational under $(\succ_B, \succ'_{-B})$ but also under $\succ'$, and therefore by uniqueness of $\phi(\succ')$ as PE and IR matching under $\succ'$, we have

$$\phi(\succ_B, \succ'_{-B}) = \phi(\succ')$$

completing the induction and the proof of Claim 2 and Theorem 10. □

- **Homework:** Find a Pareto efficient, strategyproof, but not individually rational mechanism

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- In housing markets, we can think of competitive equilibrium.

- Let $p = (p_{h_1}, ..., p_{h_n})$ be the vector of house prices.

  p_{h_i} is the price of house h_i.

- A house h_m is *affordable* for an agent i at p if $p_{h_m} \leq p_{h_i}$ (budget set).

- A matching μ and price vector p is a *competitive equilibrium*, if for any agent i, $\mu(i)$ is the best house that are affordable for i at prices p.

- In housing markets, core matching can be achieved as a competitive equilibrium.

Theorem 11 (Shapley and Scarf (1974))**.** *In each housing market, there is a competitive equilibrium that yields the core matching.*

Proof. Let $\succ$ be a preference profile and let $C_1, C_2, \ldots, C_k$ be the cycles encountered in order in Gale's TTC algorithm for this market. Let price vector p be such that for any cycle C_m and for any agent $l \in C_m$ such that $p_{h_l} = q_m$ for some constant q_m (each house in a cycle has the same price) and let $q_m > q_{m+1}$ for any $m \in \{1, 2, ..., k-1\}$ (houses in earlier cycles have higher price). Let $\bar{\mu} = \bar{\phi}(\succ)$. Observe that $(\bar{\mu}, p)$ is a competitive equilibrium: $\bar{\mu}(l)$ is an affordable house for every agent l, since $p_{\bar{\mu}(l)} = p_{h_l}$. No agent i likes some house allocated in a later cycle more than $\bar{\mu}(i)$. No agent can afford any house allocated in an earlier cycle. Hence every agent is allocated the best house she can afford. □

- **Homework:** Prove that for each housing market, the competitive equilibrium matching is unique.

  (Hint: This result is analogous to the competitive equilibrium allocation being in the core of an exchange economy.)

- For the mechanisms in housing markets, *non-bossiness* can be defined in the same way as in house allocation problems.

  A mechanism ϕ is *non-bossy*, if for any $i \in A$, for any $\succ_i, \succ_i'$ and $\succ_{-i}$,

$$\phi_i(\succ_i', \succ_{-i}) = \phi_i(\succ_i, \succ_{-i}) \implies \phi(\succ_i', \succ_{-i}) = \phi(\succ_i, \succ_{-i})$$

- **Homework:** Show that the core mechanism $\bar{\phi}$ is non-bossy.

Theorem 12. *Strategyproofness, Pareto-efficiency, and individual rationality imply non-bossiness.*

Proof. The result directly follows from Theorem 10 and the non-bossiness of the core mechanism. □

- Core mechanism can be applied to house allocation problems: randomly endow the initial houses to the agents with a uniform distribution, and then apply the core mechanism.

  This mechanism is called a *core from random endowments.*

- Core from random endowments is also a lottery mechanism, because it generates a lottery over the matchings.

- What is a relationship between the random serial dictatorship and the core from random endowments? $\Rightarrow$ Theorem 13.

Theorem 13 (Abdulkadiroğlu and Sönmez (1998))**.** *In house allocation problems, core from random endowments and random serial dictatorship are equivalent.*

3 Mixed ownership economy

House allocation problem with existing tenants

- A *house allocation problem with existing tenants* is a five-tuple $(A_E, A_N, H_O, H_V, \succ)$ where
  - A_E is a finite set of existing tenants;
  - A_N is a finite set of newcomers;
  - $H_O = \{h_i\}_{i \in A_E}$ is a finite set of occupied houses;
  - H_V is a finite set of vacant houses where $h_0 \in H_V$ denotes the null house;
  - $\succ = (\succ_i)_{i \in A_E \cup A_N}$ is a preference profile, where $\succ_i$ is a strict preference of agent $i \in A_E \cup A_N$.

    Let $\succsim_i$ be the weak-preference relation associated with $\succ_i$.

- The mechanism known as *random serial-dictatorship with squatting rights* is used in most real-life applications of these problems:

  (a) Each existing tenant decides whether she will enter the housing lottery or keep her current house. Those who prefer keeping their houses are assigned their houses. All other houses (vacant houses and houses of existing agents who enter the lottery) become available for allocation.

  (b) An ordering of agents in the lottery is randomly chosen from a given distribution of orderings. This distribution may be uniform or it may favor some groups.

  (c) Once the agents are ordered, available houses are allocated using the induced serial dictatorship: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on.

- Some examples of this mechanism include undergraduate housing at Carnegie-Mellon, Duke, Michigan, Northwestern and Pennsylvania.
- Since it does not guarantee each existing tenant a house that is as good as her own, some existing tenants may choose to keep their houses even though they wish to move, and this may result in loss of potentially large gains from trade.
- Hence, this popular mechanism is neither ex-post individually rational nor ex-post Pareto efficient.

You request my house-I get your turn (YRMH-IGYT) mechanism

- Fix an ordering of agents.
- Assign the first agent her top choice, the second agent her top choice among the remaining houses, and so on, until someone requests the house of an existing tenant.
- If at that point the existing tenant whose house is requested is already assigned another house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting the existing tenant to the top of the line and proceed with the procedure.
- Similarly, insert any existing tenant who is not already served at the top of the line once her house is requested.
- If at any point a cycle forms, it is formed by exclusively existing tenants and each of them requests the house of the tenant who is next in the cycle. (A cycle is an ordered list $(1, \ldots, k)$ of existing tenants where agent 1 demands the house of agent 2, agent 2 demands the house of agent 3, ..., agent k demands the house of agent 1.) In such cases, remove all agents in the cycle by assigning them the houses they demand and proceed.
- We denote y^f the YRMH-IGYT mechanism.

Example 2. Let

$$\begin{aligned} A_E &= \{1,2,3,4,5,6,7,8,9\} \\ A_N &= \{10,11,12,13,14,15,16\} \\ H_O &= \{h_1,h_2,h_3,h_4,h_5,h_6,h_7,h_8,h_9\} \\ H_V &= \{h_{10},h_{11},h_{12},h_{13},h_{14},h_{15},h_{16}\} \end{aligned}$$

Here h_i is the current house of existing tenant $i \in A_E$. Let the preference profile $\succ$ be given as

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h_{15}	h_3	h_1	h_2	h_9	h_6	h_6	h_6	h_{11}	h_7	h_2	h_4	h_6	h_8	h_1	h_5
$\vdots$	h_4	h_3	$\vdots$	$\vdots$	$\vdots$	h_7	h_{12}	$\vdots$	h_3	h_4	h_{14}	h_{13}	$\vdots$	$\vdots$	$\vdots$
	$\vdots$	$\vdots$				$\vdots$	$\vdots$		h_{12}	h_{16}	$\vdots$	$\vdots$			
									h_{10}	$\vdots$					
									$\vdots$						

Let the ordering of the agents be given by

$$13 - 15 - 11 - 14 - 12 - 16 - 10 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9$$

Step 1: $6 - h_6$
Step 2: $13 - h_{13}$
Step 3: $1 - h_{15}$; $15 - h_1$
Step 4: $3 - h_3$
Step 5: $4 - h_2$; $2 = h_4$
Step 6: $11 - h_{16}$
Step 7: $8 - h_8$; $14 - h_8$
Step 8: $12 - h_{14}$
Step 9: $9 - h_{11}$; $5 - h_9$; $16 - h_3$
Step 10: $7 - h_7$
Step 11: $10 - h_{10}$

Theorem 14 (Abdulkadiroğlu and Sönmez (1999)). *Any YRMH-IGYT mechanism y^f is individually rational, strategyproof, and Pareto efficient.*

- Key innovation in this mechanism is that an existing tenant whose current house is requested is upgraded to the first place at the remaining of the line before her house is allocated.

  Therefore, it is individually rational.

- A mechanism ϕ is *weakly neutral*, if labeling of vacant houses has no effect on the outcome of the mechanism.

- A mechanism ϕ is *consistent*, if the removal of a set of agents, their assignments, and a set of unassigned houses does not affect the assignments of remaining agents provided that the removal results in a well-defined reduced problem.

Theorem 15 (Sönmez and Ünver (2010))**.** *A mechanism is Pareto efficient, individually rational, strategyproof, consistent and weakly neutral if and only if it is the YRMH-IGYT mechanism* y^f*.*

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